Membership Functions II: Trends in Fuzziness and Implications

Juan Carlos Santamarina and Jean-Lou Chameau
School of Civil Engineering Purdue University West Lafayette, Indiana

ABSTRACT

Results from a set of questionnaires designed to compare methods of obtaining membership functions were analyzed to identify general trends in fuzziness. It was found that fuzziness in comprehension is always reflected in the answer; concepts such as difficulty, imprecision, and shape of membership functions are clearly interrelated; and the transition between crisp extremes through fuzziness has characteristic manifestations. Emphasis is placed on the practical implications of these ideas; applications include developing standard and type II fuzzy sets, improving the completeness and consistency of available data, and supporting the development of knowledge-based systems using fuzzy sets.

KEYWORDS: ***fuzziness, fuzzy sets, hedges, knowledge elicitation, membership functions, uncertainty***

INTRODUCTION

It is hypothesized that individuals perform as top-level observers of the complex phenomena that take place in their environment, reducing them to simpler but uncertain abstractions (see also Part I of this paper). Of concern is the extent of this transformation and the trends that characterize it. Conceptually, it is possible to analyze this question by studying the human cognitive processes of perception, storage, decision making, and response in relation to the properties of a given phenomenon. Alternatively, the combined effect of these processes can be observed indirectly by measuring the fuzziness (viewed as a dimension of uncertainty) of their response to specific tasks or stimuli.

Following this alternative approach, results from a set of questionnaires

Address correspondence to Prof. Jean-Lou Chameau, School of Civil Engineering, Purdue University, West Lafayette, IN 47907.

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distributed among graduate students and professors enrolled in a course on fuzzy sets at Purdue University were analyzed to provide insight into these issues. The questionnaires consisted of seven parts involving topics related to engineering, such as the stability of building, and others from everyday life such as the age of people. Four elicitation methods were used to determine membership functions: point estimation, interval estimation, fuzzy set exemplification, and pairwise comparison. Details on these methods of measurement and the evaluation of their applicability are given in Part I of this paper.

The study presented herein analyzes the results of the measurement methods, looks into the relationship between the difficulty of the task and the fuzziness in the result, and considers how those effects transcend the type of response mode. Ideas are further developed to show trends in fuzziness that have important consequences and potential applications.

FUZZINESS AND IMPRECISION

One of the questionnaires required individuals to evaluate the relative size (i.e., ratio of areas) of six rectangular figures, comparing them two at a time. The largest rectangle was 890 times larger than the smallest one. Figure 1 shows

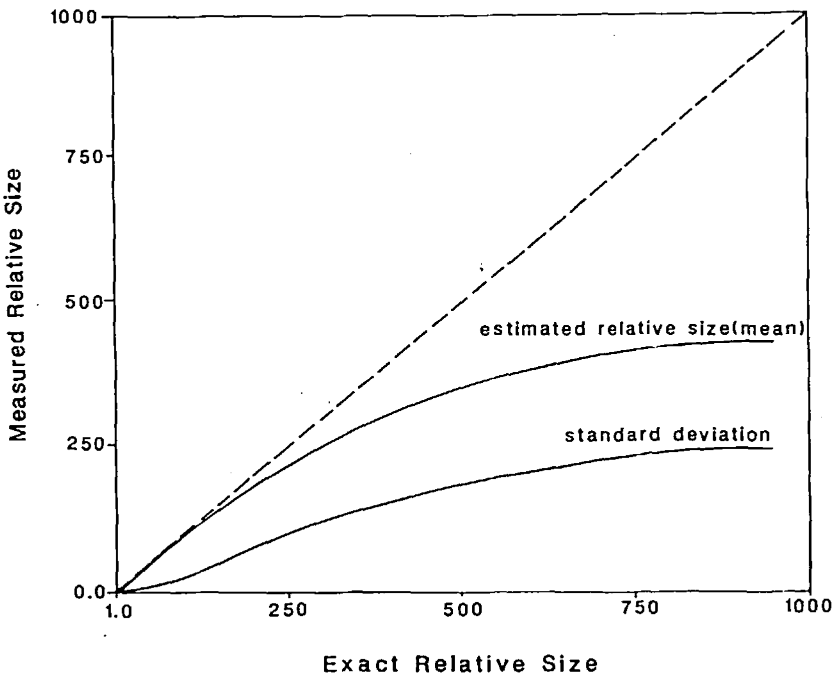

Figure 1. Estimating Relative Sizes of Figures

the mean value of the ratio estimated by the 22 assessors versus the actual ratio (plots for each individual are similar, and they all conform to Stevens's power law [1]). Figure 1 also shows the sample standard deviation (SSD) of the answers, which is a measure of discrepancy between the subjects' assessments. The second part of this questionnaire asked the subjects for an estimate of the uncertainty in their answers on a 1 to 10 scale. A plot of the mean uncertainty versus the correct size ratio is shown in Figure 2. The analysis of these figures indicates that as the degree of difficulty increases (i.e., increasing relative size of areas), the imprecision of the answers increases, as well as the discrepancy (disagreement) among assessors (Figure 1); uncertainty remains at a minimum low value when comparing elements that differ by less than one order of magnitude, and then it increases in proportion to the degree of difficulty (Figure 2). In other words, difficulty, uncertainty, imprecision, and disagreement are clearly related, all of them being minimum for the extreme simplest case of comparing two figures of the same size.

These trends were observed for all the results obtained with the four measurement techniques considered in this study, regardless of the topic of the questions. However, their manifestation differed among the elicitation methods; some of their particular characteristics are presented below.

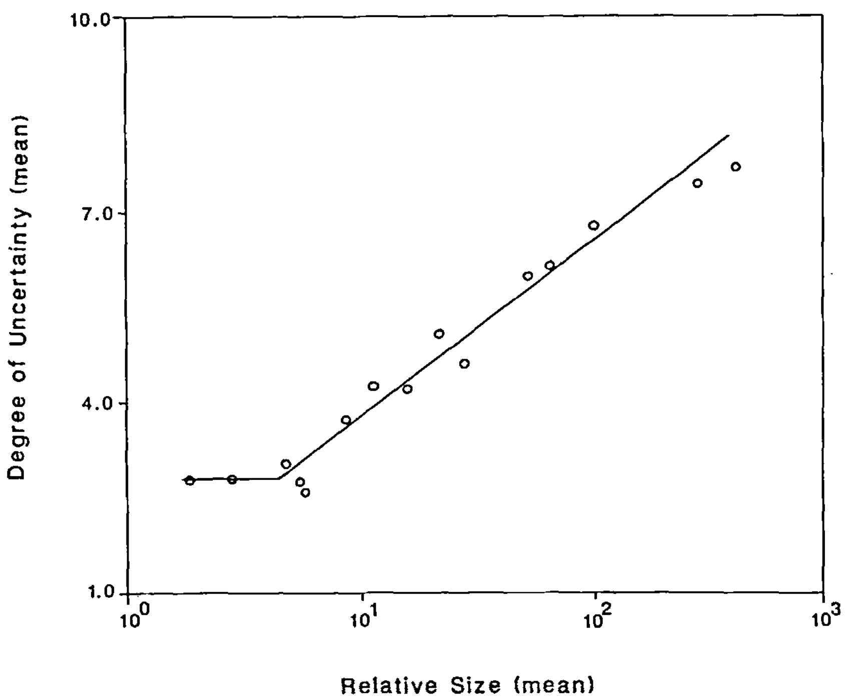

Figure 2. Uncertainty in the Estimation of Relative Sizes of Figures

FUZZINESS AND RESPONSE MODES

The pairwise comparison method consists of comparing two objects to assess the relative degree by which they possess a certain quality [2]. The range of the scale is specified, with one extreme representing the case of two objects possessing the quality with the same degree, while the other extreme corresponds to two objects that are completely different in the sense of the quality. Figure 3 shows the disagreement between assessors (expressed by the SSD of the answers) plotted against the mean relative weight for three different scales. These results are based on several different questionnaires (colors, old, tall buildings, surfaces) including data provided by Winslow [3] for the comparison of gray levels. It is observed that disagreement is maximum at the center of the scales where comparisons are the vaguest; there is perfect agreement, that is, null SSD, at the two extremes; and there is a transition between extreme situations through middle-of-scale, fuzzier comparisons.

In the point estimation response mode, individuals assign a particular object to the category it best fits according to the property under consideration. Figure 4 was developed from a questionnaire on the darkness of nine colors using a

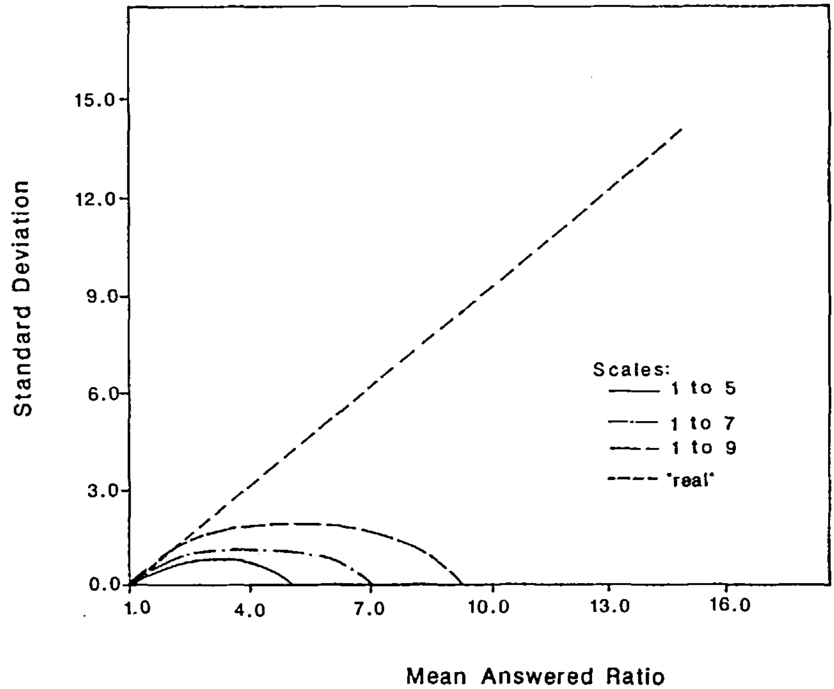

Figure 3. Pairwise Comparison—Disagreement Between Assessors

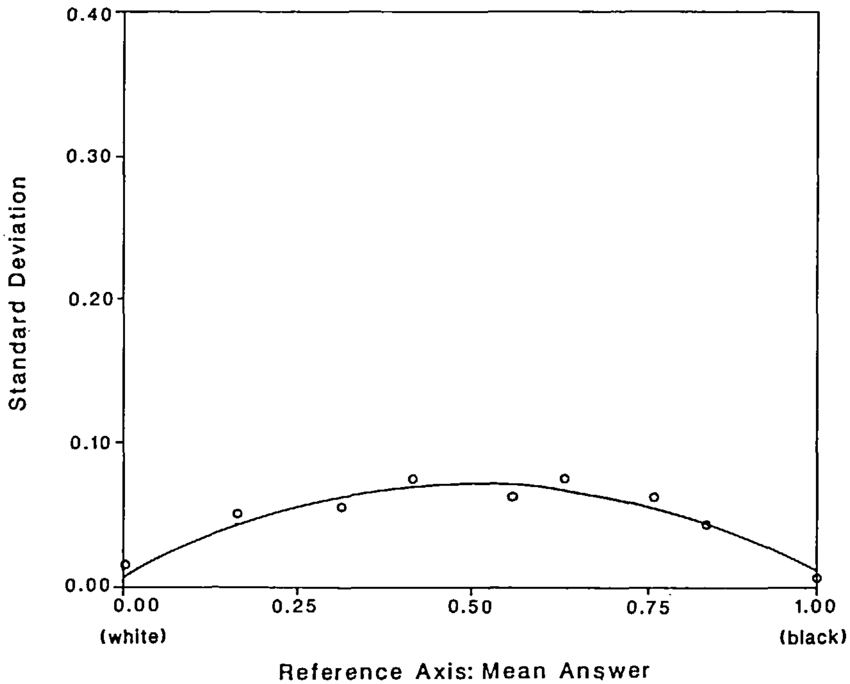

Figure 4. Point Estimation—Disagreement Between Assessors

unitary distance between any two consecutive levels of darkness to compute the SSD of the answers. Again, the trend shows maximum agreement for the answers corresponding to extreme situations, i.e., black and white colors, and maximum dispersion for intermediate ones.

In the third method, interval estimation, a segment is given as a representation of the scale of the quality being analyzed; the ends of the segment correspond to the extremes of the quality. Individuals are asked to provide the interval on such a line that best represents the object being considered. It was found that intervals close to the extremes were narrower than those intervals representing fuzzier, middle-of-scale concepts.

The fourth method evaluated with the questionnaires was the exemplification method, which consists of providing the membership values corresponding to several discrete levels selected on a reference axis. All the unimodal and monotonic membership functions derived with this method from the responses to all questionnaires were used to develop Figure 5. Agreement between assessors is observed for the crisp conditions of full support (membership value of 1.0) and null support (membership value of 0.0). Discrepancy between the answers increases toward intermediate degrees of belongingness where fuzziness is the greatest.

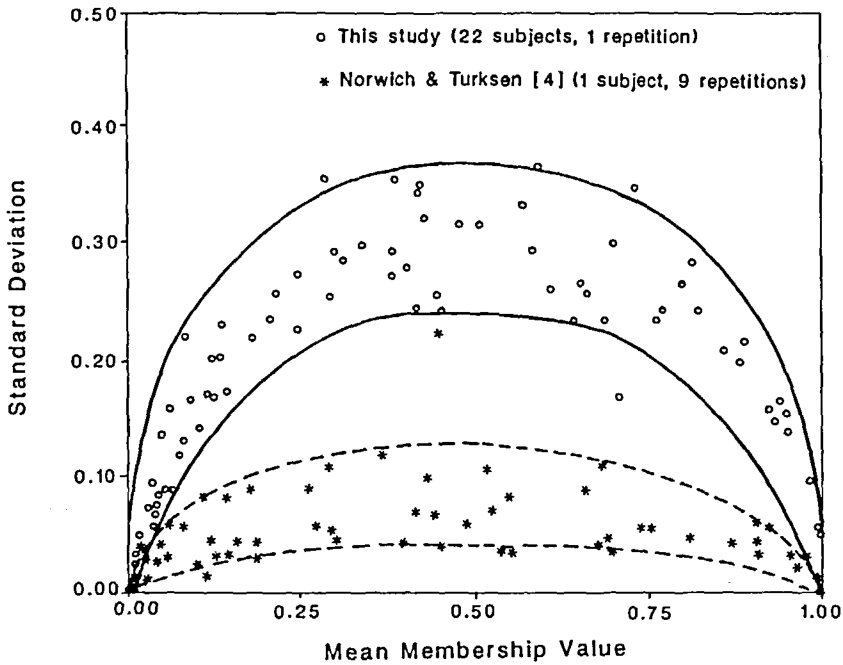

Figure 5. Membership Function Exemplification—Disagreement Between Assessors

Figure 5 also shows results by Norwich and Turksen [4, 5] for both unimodal and monotonic membership functions, but where the SSD is calculated from several answers provided by the same subject. While the two trends are the same with respect to the effect of vagueness in the transition between extremes, the important conclusion here is that the collective fuzziness of fuzziness is significantly larger than the individual one. This is relevant in the development of type II fuzzy sets and may have important implications in some knowledge acquisition problems. However, this conclusion requires further verification: The tests by Norwich and Turksen were performed under more controlled conditions than those reported here, and using different tasks.

In summary, the results presented in this section indicate that (1) fuzziness in comprehension is always transmitted to the answer, regardless of topic or response mode; and (2) the transition from one extreme (i.e., not having the quality under investigation) to the other (i.e., fully possessing the quality) is not disorganized but follows consistent trends and quantifiable relations.

FUZZINESS AND MEMBERSHIP FUNCTIONS

The study of the trends regarding the membership functions involved both unimodal and monotonic functions. For simplicity of presentation, and because the most complete data set corresponds to the method of intervals, the following discussion will use the answers obtained with this elicitation technique. Although there were minor differences in the results, these trends were the same independently of the elicitation method.

Unimodal Functions

Figure 6 shows unimodal membership functions representing the darkness of nine different colors ranging from white (No. 1) to black (No. 9). These curves are not normalized, and they are based on the answers given by the 22 subjects. It is observed that fuzziness (width) increases and maximum support (height) decreases toward the middle of the scale, that is, for colors of intermediate darkness. This characteristic of subnormality was also found by Norwich and Turksen [4], working with one subject at a time. Both extremes, black and white, are represented not by single vertical lines but by narrow fuzzy sets, showing that even "crisp" extreme conditions are not precisely perceived. These extreme membership functions are not monotonic functions formed by accumulation to the extremes, but unimodal (noncumulative) membership functions that are affected by the absolute bounds of the variable.

For each of the colors, the answers of 10 assessors were taken at random from

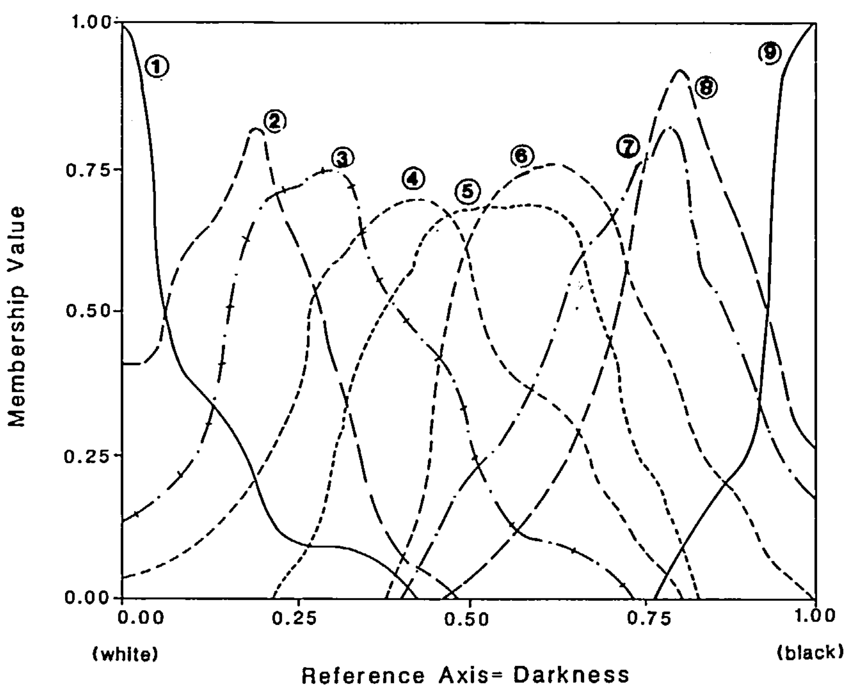

Figure 6. Unimodal Membership Functions—Transition Between Extremes

the original data, a normalized membership function was constructed, and the distance *d* between the peak point (point with a membership of 1.0) and the furthest crossover point (point with a membership of 0.5) was measured and plotted against the position of the peak point. The process was repeated nine times for each color, resulting in Figure 7. Although the approach used to develop Figure 7 is only a crude scheme to generate additional information from a limited number of data points, the observed trend is clear and very similar to those shown in Figures 3 and 4. The distance *d* is minimum at the extremes, where fuzziness is the lowest, and increases toward the center of the scale, where fuzziness is the greatest.

Figure 7 and other plots obtained from other tasks in the questionnaire have a similar shape: The central three-quarters of the curve have flatter slopes than the end parts, and the range of possible distances *d* is wider at the center of the scale than at the extremes. Therefore, major reductions in fuzziness occur close to the extremes, while the "fuzziness of fuzziness" increases toward the center of the scale. This observation is important in the construction of type II fuzzy sets. Additional information can be obtained from plots like those in Figures 3, 4, and 7: the flatter the curve, the more even the understanding of concepts across the scale; the lower the end values, the better the perception of the extremes; and the

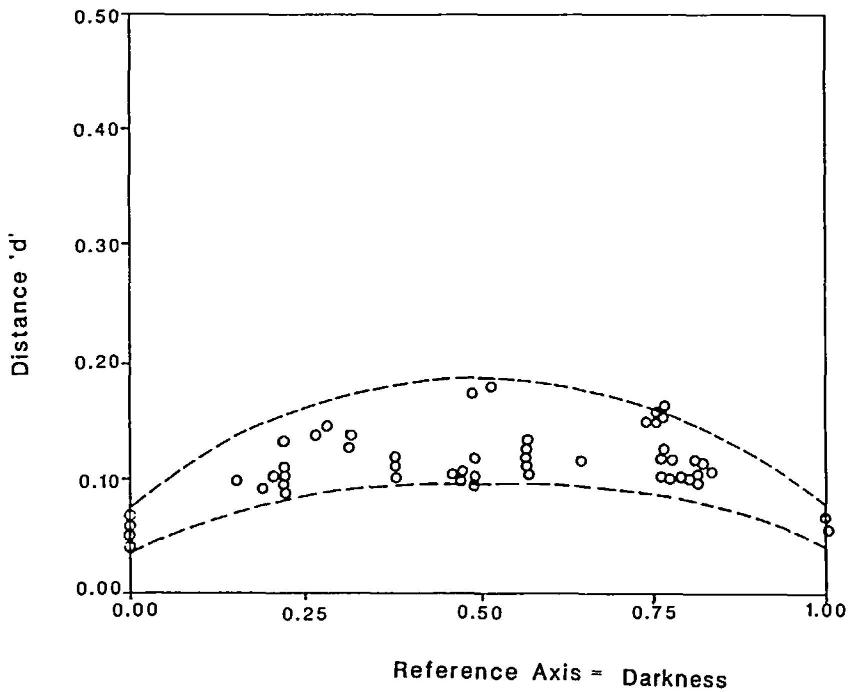

Figure 7. Unimodal Membership Functions—Width and Position

lower the curve, the more familiar or less complex the concept. All these geometric features can help characterize the fuzziness involved in comprehending a problem. A tentative classification of fuzziness based on all the results obtained in this study is shown in Figure 8 (note that the bandwidth is plotted in this figure; the bandwidth is defined as the distance between the two points with 0.5 membership value). There are five regions: extreme simplicity, low, medium, and high fuzziness, and extreme complexity. Most topics covered in this study, and to the authors' contention frequent engineering problems, fall in the category of medium fuzziness. Several applications of this classification plot will be discussed in the next sections.

Monotonic Functions

The data reported in this case (Figure 9) are the membership functions for the stability of buildings and the cumulative version of unimodal membership functions obtained for colors. The slope of each function was measured between 0.25 and 0.75 memberships and plotted against the position of the crossover point (Figure 10). It is observed that the monotonic membership functions are steepest at the extremes and flattest in the central part of the scale, with a fairly

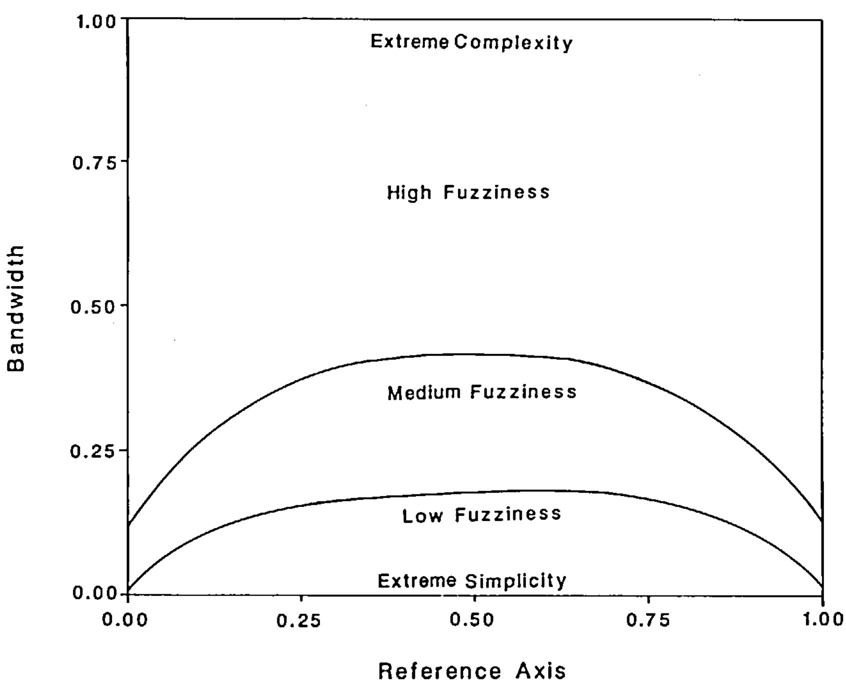

Figure 8. Unimodal Membership Functions—Classification of Fuzziness

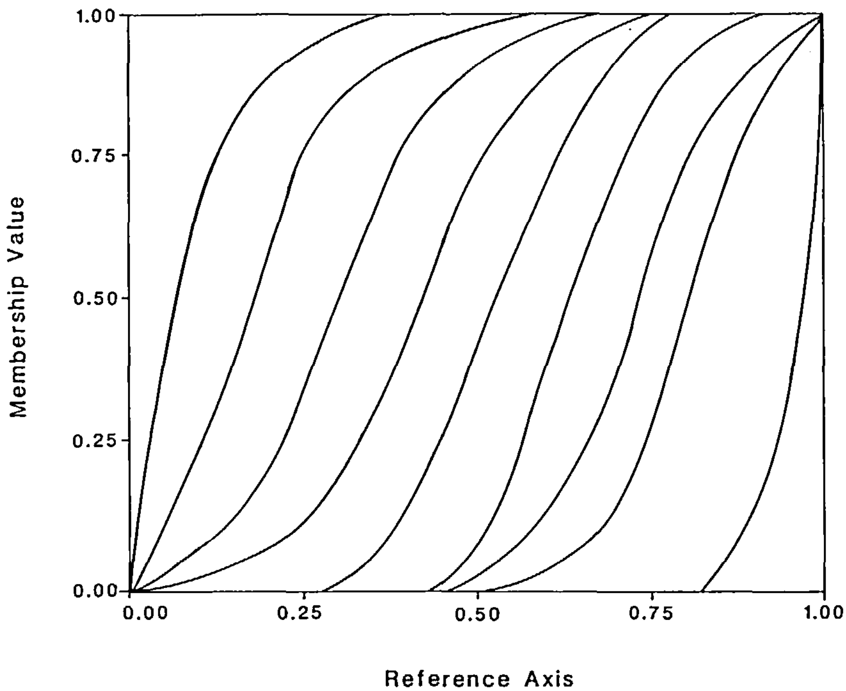

Figure 9. Monotonic Membership Functions—Transition Between Extremes

symmetrical trend about the middle of the scale. The relationship between the absolute value of steepness and fuzziness is very stable. These is no value plotted close to the extremes because, as mentioned earlier, bounds also have fuzzy representation, and thus the crossover points are away from the ends. The characterization of problems according to their degree of fuzziness proposed in Figure 8 for the case of unimodal membership functions could be repeated for monotonic ones using the slope of the functions.

HEDGES—EXTREMES—EXPERIENCE

Figure 10 clearly shows the effect of linguistic intensifiers (hedges): The flatness in the central part of the scale is consistent with the translation effect of hedges found by several researchers [4, 6, 7]. However, for situations closer to the ends of the scale, the trend in Figure 10 supports the steepening effect of hedges suggested by Zadeh [8]. Therefore, it is not a matter of which approach is correct but of where the membership function lies within the range of the variable.

It was observed that, even for the obvious case of darkness, the extremes

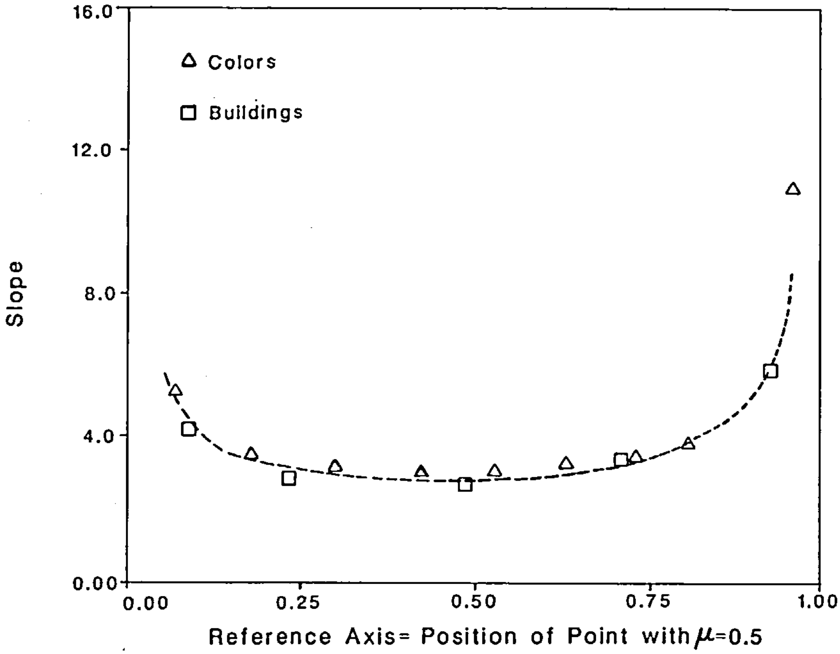

Figure 10. Monotonic Membership Functions—Slope and Position

black and white did not have crisp representations (Figure 6). This indicates that for most practical situations extremes will be fuzzy. Therefore, if their definition is needed, it is the analyst's responsibility to establish it for the particular problem under consideration. When results are obtained from repeated measures from the same subject performing as measurement instrument, fuzziness in the extremes can be used as a measure of the fuzziness of threshold and saturation levels of the subject's perception. If aggregation or averaging of the results from different assessors is used to seek a consensus, one should expect higher fuzziness because of the variability in perception among assessors.

The existence of both extremes of a quality is a key concept in this study: knowing the extremes of the variables involved is equivalent to fixing the context and all its related valuation scales (in communication, for example, this is a fundamental step in order to share meaning). The validity of Figures 8 and 10 for a wide variety of tasks indicates that once the individual knows the context and its extremes, he or she can act within it and make inferences without the need for the objective measures (valuation scales) of the variables that govern the environment. In other words, the following two fuzzy propositions:

- T is very tall
- P is shorter but not much

conduce to "P is quite tall." This inference was made without specifying a scale for height, and it is as valid in this world as in Lilliput. Its translation to values of height is immediate once the context (extremes of "height") is known. Note that the example could have been related to earthquake accelerations and still be valid, and "quite high" would still keep the same relative position. Furthermore, the real value of the acceleration is not needed to know that "quite high accelerations" produce important damage; however, this does not imply that the value space is not relevant—in fact, it was part of the experiential background used by the subject to set the context.

The value meaning of the qualitative statements depends on the individual's previous experience, implying that subjects perform as holders of "adaptive molds" that are modified through a cycle of experience-feedback-adaptation as they better comprehend the phenomenon. From this perspective, the position of an experimental curve in Figure 8 is indicative of the individual's skill level: the higher the curve, the lower the previous experience [9].

POTENTIAL APPLICATIONS

There are many potential applications of these concepts. For example, it is possible to reverse the process used in developing Figures 8 and 10: Given the estimated position of a stimulus on a segment bounded by the extremes of the variable, one can enter these plots and obtain the width of unimodal functions or the slope of monotonic ones; then functions like pi, s, and z curves can be used to obtain the membership values. Implicit to the application of this simplified procedure is the need to recognize the extremes of the variable, select an appropriate level of fuzziness, and assume that the scale is linear.

If the perception of the variable does not follow a linear scale, the assessor must also identify intermediate points, for example by "bisection." A modified version of Stevens's power law of psychophysics [1] can also be used: The assumption is that subjects tend to map an input into a linear scale when they are allowed to respond in linguistic form. Then Stevens's law can be expressed as

$$L = \log c + a' \log I$$

where L is the linguistic scale, c is a constant, a' is an exponent dependent upon stimulus involved, and I is the input. Data from questionnaires used in this study, and others related to the field of geotechnical engineering, were analyzed to verify this hypothesis. It was found that the modified law is supported by individual measurements and also by the average of collective measures. Table 1 lists values of a' that were determined from the data. Among all the cases considered in Table 1, the coefficient a relating surprise and probability of failure was found to be strongly case- and individual-dependent.

Besides this simplified approach to developing membership functions,

Table 1. Modified Power Law: Linguistic Response

Stimulus	Exponent a'
Stability of buildings (H/B)	0.87
N (Standard Penetration Test)	0.62
Sensitivity of clays	0.81
Strength of clays	0.65
Earthquake acceleration	0.5
Depth for soil improvement	0.52
Time for soil improvement	0.32
Permeability of soils	0.12
Probability of failure (surprise)	−0.62

concepts discussed in this paper can be applied:

- To develop "probabilistic" and "type II" fuzzy sets (aided by the trends shown in Figures 5, 8, and 10).
- To determine the level of a subject's understanding of a problem.
- To fuzzify singletons.
- To check, correct, or improve results based on incomplete, limited, or inconsistent data.
- To improve quality and validity of simple examples, such as those often used to illustrate fuzzy set applications.

Andonyadis [10] used the findings of this study to improve incomplete and inconsistent data in the development of a pavement management system based on fuzzy set theory. An example of the fifth suggestion is the development of membership functions for linguistic matching based on the simplified technique described above. Finally, the concept that one does not need support values but just a list of degrees of belief to implement fuzzy operations and fuzzy inferences in a given context and the idea of transition between extremes have been used to develop a powerful fuzzy shell-environment in LISP [11].

CONCLUSIONS

A questionnaire designed to determine the advantages and limitations of several elicitation techniques (Part I of this paper) also helped define and evaluate the characteristics of fuzziness in comprehension. The main conclusions of this study are:

1. There is a clear interrelation among: difficulty, uncertainty, imprecision, dispersion between assessors, width of estimated intervals, height and width of unimodal membership functions, and steepness of monotonic functions.

2. The fuzziness of the questions is always reflected in the answer, regardless of the response mode.
3. There is a gradual transition from crisp extreme situations to fuzzy intermediate ones. This trend is symmetrical and shows most of the reduction in fuzziness occurring close to the extremes, leaving an intermediate region of a uniform degree of fuzziness.
4. The strength and general validity of the observed trends, for all topics considered, indicate the possibility of modeling fuzzy inferences just with a list of degrees of belief (without the support values) once a context is agreed upon.
5. Even "clear" extreme situations are not perfectly perceived by assessors and have fuzzy representations.
6. Recognition of these trends may help in the application of fuzzy set mathematics in several ways, such as in developing membership functions, rating fuzziness and skill, developing type II fuzzy sets, and evaluating incomplete or inconsistent data. Additional findings include a rationale for the effects of hedges, confirming the importance of shift as suggested by previous investigators.

ACKNOWLEDGMENTS

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